

# Elliptic Operators in Subspaces and the Eta Invariant

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February 1, 2008

## Abstract

In this paper we obtain a formula for the fractional part of the  $\eta$ -invariant for elliptic self-adjoint operators in topological terms. The computation of the  $\eta$ -invariant is based on the index theorem for elliptic operators in subspaces obtained in [SS99], [SS00b]. We also apply the  $K$ -theory with coefficients  $\mathbb{Z}_n$ . In particular, it is shown that the group  $K(T^*M, \mathbb{Z}_n)$  is realized by elliptic operators (symbols) acting in appropriate subspaces.

**Keywords:** index of elliptic operators in subspaces,  $K$ -theory, eta-invariant, mod  $k$  index, Atiyah–Patodi–Singer theory

**2000 AMS classification:** Primary 58J28, Secondary 58J20, 58J22, 19K56

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\*Supported by DAAD via IQN “Kopplungsprozesse und ihre Strukturen in der Geo- und Biosphäre”, by DFG via a project with the Arbeitsgruppe “Partielle Differentialgleichungen und Komplexe Analysis,” Universität Potsdam, by RFBR grants Nos. 02-01-00118, 02-01-06515 and 00-01-00161, and by the European Training Network Program “Geometric Analysis”.

# Introduction

The spectral  $\eta$ -invariant of an elliptic self-adjoint operator on a closed manifold was introduced by Atiyah, Patodi, and Singer [APS75]. It appeared as a nonlocal contribution to an index formula for manifolds with boundary obtained via the heat equation method. From the very moment of its introduction, it was clear that this spectral invariant in the general case is neither an invariant of the principal symbol of the operator nor a homotopy invariant of the operator itself. More precisely, for a one-parameter family  $A_t$  of elliptic self-adjoint operators, the function  $\eta(A_t)$  is *piecewise smooth* with jumps at those values of  $t$ , where some eigenvalue of the operator in the family changes its sign.

P. Gilkey [Gil89] observed that for *differential operators* satisfying the parity condition

$$\text{ord} A + \dim M \equiv 1 \pmod{2}, \quad (1)$$

the  $\eta$ -invariant of a one-parameter family is a *piecewise constant* function. In particular, in this case the fractional part of the spectral  $\eta$ -invariant is in fact a homotopy invariant depending on the principal symbol of the operator only. This rises the problem of computing this invariant in topological terms and the problem of finding nontrivial geometric examples.

The situation is rather well understood for even-dimensional manifolds. In this case the famous first order Dirac type operators satisfy the parity condition (1). These were studied by P. Gilkey [Gil85]. He proved that the  $\eta$ -invariant takes dyadic rational values. Nontrivial  $\eta$ -invariants were computed on some nonorientable  $\text{pin}^c$  manifolds, e.g.  $\mathbb{RP}^{2n}$ . This fractional invariant is important in topology and differential geometry (e.g., see [Sto88, BG87, Gil98]).

On odd-dimensional manifolds, P. Gilkey showed in [Gil89] that the fractional part of the  $\eta$ -invariant defines a homomorphism

$$K(P^*M)/K(M) \longrightarrow \mathbb{Z} \left[ \frac{1}{2} \right] / \mathbb{Z},$$

where  $P^*M = S^*M/\mathbb{Z}_2$  denotes the projective space bundle of  $M$ . Moreover, he introduced a class of second order operators on  $M$  that define nontrivial elements of the latter  $K$ -group and proposed a problem of computing their  $\eta$ -invariants.

The aim of the present paper is to obtain a topological formula for the fractional part of the  $\eta$ -invariant for operators satisfying the parity condition.

The  $\eta$ -invariant of an operator  $A$  satisfying condition (1) is completely determined by the nonnegative spectral subspace  $\widehat{L}_+(A)$  of this operator, while the fractional part of the invariant is determined by the so-called *symbol of the subspace*.

This is a vector bundle on the cospheres  $S^*M$  over the manifold generated by the positive eigenspaces of the self-adjoint symbol  $\sigma(A)$ . First, this enables one to identify the  $\eta$ -invariant of self-adjoint elliptic operators with a dimension-type functional (see [SS99]) on the corresponding (infinite-dimensional) spectral subspaces. Second, we can apply the index formula for elliptic operators in subspaces [SS99, SS00b]. The index formula reduces the computation to the "index modulo  $n$ " problem for operators in subspaces. The term "modulo  $n$ " here expresses the fact that in this case the index of an elliptic operator, being reduced modulo  $n$ , becomes an invariant of the principal symbol of the operator. It turns out that such elliptic operators on a closed manifold define the  $K$ -theory with coefficients in  $\mathbb{Z}_n$ . In particular, the index is computed modulo  $n$  by the direct image mapping in  $K$ -theory.

The fractional part of the  $\eta$ -invariant was first computed in the classical Atiyah–Patodi–Singer paper [APS76] for operators with coefficients in flat bundles. However, their result does not apply to our operators, since there is no flat bundle available. Let us also mention that although we do not rely on the results of [APS76], there are strong parallels between the two proofs. For instance, the index formula in subspaces plays the same role as the Atiyah–Patodi–Singer index formula for trivialized flat bundles, while the mod  $n$ -index formula of the present paper generalizes the mod  $n$ -spectral flow formula.

**Added January 2002.** The main formula for the fractional part of the  $\eta$ -invariant was used to find *nontrivial*  $\eta$ -invariants for some new second order geometric operators in the papers [SS00a, SS01]. Thus, the formula for the fractional part of the  $\eta$ -invariant on odd-dimensional manifolds is not empty and the problem of finding even-order operators in odd dimensions with nontrivial  $\eta$ -invariant is solved.

The results of the paper were reported at the international conferences "Operator Algebras and Asymptotics on Manifolds with Singularities," Warsaw, and "Jean Leray 99", Karlskrona, Sweden. We are thankful to the participants of Professor A. S. Mishchenko seminar at the Moscow State University for useful discussions. We would like to thank the referees for their critical remarks on the preliminary version of the paper.

## 1 Subspaces and index formulas

**1. Pseudodifferential subspaces. The dimension functional.** Spaces defined by pseudodifferential projections on a smooth closed manifold  $M$  were considered in [SS99, SS00b]. More precisely, a subspace

$$\widehat{L} \subset C^\infty(M, E)$$

in the space of smooth sections of a vector bundle  $E$  on  $M$  is said to be *pseudodifferential* if it is the range

$$\widehat{L} = \text{Im } P, \quad P : C^\infty(M, E) \rightarrow C^\infty(M, E)$$

for some pseudodifferential projection  $P$  of order zero. The principal symbol of the projection defines the vector bundle

$$L = \text{Im } \sigma(P) \subset \pi^*E \in \text{Vect}(S^*M). \quad (2)$$

It is the *symbol of the subspace*. Here  $\pi : S^*M \rightarrow M$  is the projection for the cosphere bundle.

On the cotangent bundle  $T^*M$ , we consider the antipodal involution

$$\alpha : T^*M \longrightarrow T^*M, \quad \alpha(x, \xi) = (x, -\xi).$$

A subspace  $\widehat{L} \subset C^\infty(M, E)$  is said to be *even (odd)* with respect to  $\alpha$  if the symbol  $L$  is invariant (antiinvariant):

$$L = \alpha^*L, \quad \text{or} \quad L \oplus \alpha^*L = \pi^*E. \quad (3)$$

We point out that both equalities in this formula are equalities of subbundles in the ambient bundle  $\pi^*E$ . Denote the semigroups of even(odd) subspaces by  $\widehat{\text{Even}}(M)$  ( $\widehat{\text{Odd}}(M)$ ). The symbols of even (odd) subspaces will be referred to as even (odd) bundles for brevity.

It turns out that if the parities of the subspaces and of the dimension of  $M$  are opposite, then the subspaces have a homotopy invariant similar to the dimension for finite-dimensional vector spaces. More precisely, the following theorem holds.

**Theorem 1.** [SS99, SS00b] *There is a unique additive functional*

$$d : \widehat{\text{Even}}(M^{\text{odd}}) \rightarrow \mathbb{Z} \left[ \frac{1}{2} \right], \quad \text{or} \quad d : \widehat{\text{Odd}}(M^{\text{ev}}) \rightarrow \mathbb{Z} \left[ \frac{1}{2} \right]$$

*with the following properties:*

1. (*invariance*)  $d(U\widehat{L}) = d(\widehat{L})$  for all invertible pseudodifferential operators  $U$  with even principal symbol:  $\alpha^*\sigma(U) = \sigma(U)$ ;
2. (*relative index*)  $d(\widehat{L}_1) - d(\widehat{L}_2) = \text{ind}(\widehat{L}_1, \widehat{L}_2)$  for two subspaces with coinciding principal symbols;<sup>1</sup>

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<sup>1</sup>The relative index of subspaces  $\text{ind}(\widehat{L}_1, \widehat{L}_2)$  is expressed via the Fredholm index [BDF77]

$$\text{ind}(\widehat{L}_1, \widehat{L}_2) = \text{ind}(P_2 : \text{Im } P_1 \rightarrow \text{Im } P_2),$$

3. (complement)  $d\left(\widehat{L}\right) + d\left(\widehat{L}^\perp\right) = 0$ , where  $\widehat{L}^\perp$  denotes the orthogonal complement of  $\widehat{L}$ .

**Corollary 1.** *The functional  $d$  is a homotopy invariant of the subspace, while its fractional part is an invariant of the symbol of the subspace.*

Indeed, the homotopy invariance follows from the invariance property. Moreover, it follows from the relative index property that the fractional part is determined by the symbol of the subspace.

**2. Dimension functional and  $\eta$ -invariant.** The functional  $d$  can be expressed in terms of the Atiyah–Patodi–Singer  $\eta$ -invariant.

Namely, for an elliptic self-adjoint operator

$$A : C^\infty(M, E) \longrightarrow C^\infty(M, E)$$

of a positive order consider the subspace  $\widehat{L}_+(A) \subset C^\infty(M, E)$  generated by the eigenvectors of  $A$  corresponding to nonnegative eigenvalues. It is well known (e.g., see [APS75]) that the spectral projection  $P_+(A)$  on this subspace is a pseudodifferential operator of order zero. Thus, the subspace  $\widehat{L}_+(A)$  is pseudodifferential. The symbol  $L_+(A)$  of the subspace can be explicitly calculated:

$$L_+(A) = \text{Im } \sigma(P_+(A)) \subset \pi^*E \in \text{Vect}(S^*M),$$

where the principal symbol  $\sigma(P_+(A))$  of the projection is equal to the spectral projection for the principal symbol  $\sigma(A)$ :  $\sigma(P_+(A)) = P_+(\sigma(A))$ .

Thus, if  $A$  is a differential operator, then the subspace  $\widehat{L}_+(A)$  is either even or odd, according to the parity of operator's order. The same property holds for a class of pseudodifferential operators introduced in [Gil89]: these are classical pseudodifferential operators with homogeneous terms in the asymptotic expansion of the symbol possessing the  $\mathbb{R}_*$ -invariance (cf. [NSS92]):

$$\sigma(A)(x, \xi) \sim \sum_{j=0}^{\infty} a_{d-j}(x, \xi), \quad a_k(x, -\xi) = (-1)^k a_k(x, \xi), \quad \text{for all } k \leq d. \quad (4)$$

For this class of operators, the functional  $d$  is equal to the  $\eta$ -invariant.

**Theorem 2.** [SS99, SS00b] *For the nonnegative spectral subspace  $\widehat{L}_+(A)$  of an elliptic self-adjoint operator  $A$  satisfying (4) one has*

$$d\left(\widehat{L}_+(A)\right) = \eta(A) \quad (5)$$

*provided the order of  $A$  and the dimension of the manifold have opposite parities.*

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where the projections  $P_{1,2}$  define  $\widehat{L}_{1,2}$ .

According to this result, to compute the fractional part of the  $\eta$ -invariant it suffices to compute the fractional part of the functional  $d$  in terms of the symbol of the subspace. An important ingredient of the computation is the index formula for elliptic operators in subspaces.

**3. Elliptic theory in subspaces.** Let  $\widehat{L}_{1,2} \subset C^\infty(M, E_{1,2})$  be two subspaces. Consider a pseudodifferential operator

$$D : C^\infty(M, E_1) \longrightarrow C^\infty(M, E_2)$$

in the ambient spaces. If it preserves the subspaces:  $D\widehat{L}_1 \subset \widehat{L}_2$ , then the restriction

$$D : \widehat{L}_1 \longrightarrow \widehat{L}_2 \tag{6}$$

is called an *operator acting in subspaces*. In this case the principal symbol  $\sigma(D)$  restricts to a vector bundle homomorphism

$$\sigma(D) : L_1 \longrightarrow L_2 \tag{7}$$

over  $S^*M$ . This is called the *symbol of the operator in subspaces*.

It is proved in [SSS98] that the closure

$$D : H^s(M, E_1) \supset \widehat{L}_1 \longrightarrow \widehat{L}_2 \subset H^{s-m}(M, E_2), \quad m = \text{ord} D,$$

of (6) in the Sobolev norms defines a Fredholm operator if the symbol (7) is *elliptic*, i.e., a vector bundle isomorphism.

For elliptic operators the following index formula was obtained in [SS99, SS00b].

**Theorem 3.** *One has*

$$\text{ind}(D, \widehat{L}_1, \widehat{L}_2) = \frac{1}{2} \text{ind} \widetilde{D} + d(\widehat{L}_1) - d(\widehat{L}_2), \tag{8}$$

where  $D : \widehat{L}_1 \longrightarrow \widehat{L}_2$ ,  $\widehat{L}_{1,2} \subset C^\infty(M, E_{1,2})$  is an elliptic operator in subspaces of the same parity:  $\widehat{L}_{1,2} \in \widehat{\text{Even}}(M^{\text{odd}})$  or  $\widehat{\text{Odd}}(M^{\text{ev}})$ , while elliptic operator

$$\widetilde{D} : C^\infty(M, E_1) \longrightarrow C^\infty(M, E_2)$$

has principal symbol

$$\sigma(\widetilde{D}) = \sigma(D) \oplus \alpha^* \sigma(D) : L_1 \oplus \alpha^* L_1 \longrightarrow L_2 \oplus \alpha^* L_2$$

for odd subspaces. In the case of even subspaces,  $\widetilde{D}$  is defined as

$$\begin{aligned} \widetilde{D} &: C^\infty(M, E_1) \longrightarrow C^\infty(M, E_1), \\ \sigma(\widetilde{D}) &= [\alpha^* \sigma(D)]^{-1} \sigma(D) \oplus 1 : L_1 \oplus L_1^\perp \longrightarrow L_1 \oplus L_1^\perp. \end{aligned}$$

## 2 Computation of the $\eta$ -invariant

**Proposition 1.** *For a subspace  $\widehat{L} \in \widehat{\text{Even}}(M^{\text{odd}})$  or  $\widehat{\text{Odd}}(M^{\text{ev}})$  with symbol  $L$ , there exists a positive integer  $N$  such that the direct sum  $2^N L$  on  $S^*M$  can be lifted from the base  $M$ . That is, for some vector bundle  $F \in \text{Vect}(M)$  there exists an isomorphism*

$$\sigma : 2^N L \longrightarrow \pi^* F, \quad 2^N L = \underbrace{L \oplus \dots \oplus L}_{2^N \text{ copies}}. \quad (9)$$

*Proof.* The part of the theorem, pertaining to even subspaces, follows from [Gil89], where it is shown that for an odd-dimensional  $M$  the projective space bundle

$$P^*M = S^*M / \{(x, \xi) \sim (x, -\xi)\}$$

has the same  $K$ -theory groups as  $M$ , except for the 2-torsion. The isomorphism, modulo 2-torsion, is established by the natural projection  $\pi_P : P^*M \rightarrow M$ . More precisely,  $\ker \pi_P^* = 0$ , and  $\text{coker} \pi_P^*$  is a 2-torsion group.

On the other hand, it is shown in [SS00b] that for an odd vector bundle  $L$  on  $S^*M$  and  $N$  large enough the bundle  $2^N L$  is isomorphic to its complement  $2^N \alpha^* L$ :

$$\sigma_0 : 2^N L \xrightarrow{\simeq} 2^N \alpha^* L$$

(this can be obtained noting that the projection  $S^*M \rightarrow P^*M$  for even-dimensional manifolds induces an isomorphism in  $K$ -theory, modulo 2-torsion). Now we can construct the desired pull-back (9) by the formula

$$\sigma : 2^{N+1} L \xrightarrow{1 \oplus \sigma_0} 2^N L \oplus 2^N \alpha^* L = 2^N \pi^* F.$$

□

Let us now consider an elliptic operator in subspaces

$$\widehat{\sigma} : 2^N \widehat{L} \longrightarrow C^\infty(M, F)$$

with symbol (9). We write out the index formula for this operator:

$$\text{ind} \left( \widehat{\sigma}, 2^N \widehat{L}, C^\infty(M, F) \right) = \frac{1}{2} \text{ind} \widehat{\sigma} + 2^N d \left( \widehat{L} \right). \quad (10)$$

This formula, along with the integrality of the index, implies that the functional  $d$  is dyadic rational and has at most  $2^{N+1}$  as the denominator. For the fractional part of  $d$  this gives

$$\left\{ d \left( \widehat{L} \right) \right\} = \frac{1}{2^N} \left( \text{mod } 2^N \text{-ind} \left( \widehat{\sigma}, 2^N \widehat{L}, C^\infty(M, F) \right) - \frac{1}{2} \left( \text{mod } 2^{N+1} \text{-ind} \widehat{\sigma} \right) \right).$$

In particular, for  $N = 0$  this gives a topological formula for the fractional part of  $d(\widehat{L})$ . For  $N \geq 1$  it remains to compute the index modulo  $2^N$  for an elliptic operator in subspaces

$$\widehat{\sigma} : 2^N \widehat{L} \longrightarrow C^\infty(M, F).$$

This problem is solved in the next section.

### 3 Index theory modulo $n$

For a given positive integer  $n \geq 2$ , we consider elliptic operators in subspaces of a special form:

$$D : n\widehat{L} \longrightarrow C^\infty(M, F). \quad (11)$$

Let us emphasize that the subspace  $\widehat{L}$  need not satisfy the parity condition. It is easy to show that the index of  $D$  modulo  $n$  denoted by

$$\text{mod } n\text{-ind } D \in \mathbb{Z}_n$$

is determined by the principal symbol  $\sigma(D) : nL \longrightarrow \pi^*F$ .

It is natural to compute this index with values in  $\mathbb{Z}_n$ , in terms of a difference construction with values in  $K$ -theory with  $\mathbb{Z}_n$  coefficients:

$$[\sigma(D)] \in K(T^*M, \mathbb{Z}_n).$$

The necessary information about this theory is provided for the reader's convenience in the Appendix.

Let us define this difference construction. First of all, we rewrite the group  $K(T^*M, \mathbb{Z}_n)$  in terms of the usual  $K$ -theory. We have

$$K(T^*M, \mathbb{Z}_n) = K(T^*M \times \mathbb{M}_n, T^*M \times pt), \quad (12)$$

where  $\mathbb{M}_n$  is the so-called *Moore space*. It readily follows from (12) that the elements of  $K(T^*M, \mathbb{Z}_n)$  can be realized as families of elliptic symbols<sup>2</sup> on the manifold  $M$ ,

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<sup>2</sup>Here we use the natural construction [AS71] that assigns an element

$$[\sigma] \in K(T^*M \times X)$$

of the  $K$ -group to each family  $\sigma(x)$ ,  $x \in X$ , of elliptic symbols on the manifold  $M$  with the parameter space  $X$ :

$$\sigma(x) : \pi^*E \longrightarrow \pi^*F, \quad E, F \in \text{Vect}(M \times X), \quad \pi : S^*M \times X \rightarrow M \times X.$$



where the Moore space serves as the parameter space for the family. Thus, the desired family of elliptic symbols is defined as the composition of elliptic families in subspaces:

$$[\sigma(D)] = \left[ \pi^* F \xrightarrow{\sigma^{-1}(D)} nL \xrightarrow{\beta^{-1} \otimes 1_L} \gamma_n \otimes nL \xrightarrow{1_\gamma \otimes \sigma(D)} \gamma_n \otimes \pi^* F \right] \quad (13)$$

$$[\sigma(D)] \in K(T^*M \times \mathbb{M}_n, T^*M \times pt),$$

where  $\gamma_n$  is the line bundle corresponding to the generator of the reduced group  $\tilde{K}(\mathbb{M}_n) \simeq \mathbb{Z}_n$  and  $\beta$  is a trivialization  $\beta : n\gamma_n \rightarrow \mathbb{C}^n$ .

**Theorem 4.** (*index theorem modulo  $n$* )

$$\text{mod } n\text{-ind } D = p_! [\sigma(D)], \quad (14)$$

where the direct image in  $K$ -theory (with coefficients)

$$p_! : K(T^*M, \mathbb{Z}_n) \longrightarrow K(pt, \mathbb{Z}_n) = \mathbb{Z}_n,$$

is induced by the mapping  $p : M \longrightarrow pt$ .

*Proof.* Consider the following three families of elliptic operators in subspaces, parametrized by the Moore space  $\mathbb{M}_n$

$$\begin{aligned} C^\infty(M, F) &\xrightarrow{D^{-1}} n\hat{L}, \\ n\hat{L} &\xrightarrow{\beta^{-1} \otimes 1_{\hat{L}}} \gamma_n \otimes n\hat{L}, \\ \gamma_n \otimes n\hat{L} &\xrightarrow{1_\gamma \otimes D} \gamma_n \otimes C^\infty(M, F) \end{aligned}$$

(here  $D^{-1}$  denotes an almost inverse, i.e. inverse up to compact operators, of  $D$  and the three families correspond to the symbols in (13)). The first family is constant. The second family is defined by isomorphisms, while the third family is merely the tensor product of  $D$  and the bundle  $\gamma_n$  over the parameter space. Hence, the index of the composition is equal to

$$\text{ind}([1_\gamma \otimes D] \circ [\beta^{-1} \otimes 1_{\hat{L}}] \circ D^{-1}) = [\gamma_n] \text{ind} D + 0 - \text{ind} D \in K(\mathbb{M}_n). \quad (15)$$

On the other hand, the index of the elliptic family

$$[1_\gamma \otimes D] \circ [\beta^{-1} \otimes 1_{\hat{L}}] \circ D^{-1} : C^\infty(M, F) \longrightarrow \gamma_n \otimes C^\infty(M, F)$$

is calculated by the Atiyah-Singer index formula for families (see [AS71]). Thus, by virtue of (13), this gives

$$\text{ind}([1_\gamma \otimes D] \circ [\beta^{-1} \otimes 1_{\hat{L}}] \circ D^{-1}) = p_! [\sigma(D)] \in K(\mathbb{M}_n). \quad (16)$$

On the other hand, taking into account the isomorphism  $K(\mathbb{M}_n) = \mathbb{Z} \oplus \mathbb{Z}_n$  with  $[\gamma_n] - 1$  as a generator of the torsion part  $\mathbb{Z}_n$ , we obtain

$$\text{mod } n\text{-ind } D = p! [\sigma(D)]$$

comparing (16) with (15). □

**Remark 1.** A similar mod  $n$ -index theorem for boundary value problems was obtained in [Fre88], [FM92].

## 4 A formula for the fractional part

In this section we write out the final formula for the  $\eta$ -invariant. Namely, for a subspace  $\widehat{L} \subset C^\infty(M, E)$  and the pull-back of its symbol from  $M$  by an isomorphism  $\sigma$

$$2^N L \xrightarrow{\sigma} \pi^* F, \quad F \in \text{Vect}(M), \pi : S^*M \rightarrow M,$$

in Section 2 we expressed the fractional part of  $d(\widehat{L})$  as

$$\left\{ d(\widehat{L}) \right\} = \frac{1}{2^N} \left( \text{mod } 2^N\text{-ind} \left( \widehat{\sigma}, 2^N \widehat{L}, C^\infty(M, F) \right) - \frac{1}{2} \left( \text{mod } 2^{N+1}\text{-ind} \widehat{\widehat{\sigma}} \right) \right).$$

The two terms, in fact, can be combined together. Namely, in the even (odd) cases the resulting formulas are, respectively,

$$\begin{aligned} \left\{ d(\widehat{L}) \right\} = & \\ & \frac{1}{2^{N+1}} \text{mod } 2^{N+1}\text{-ind} \left( 2^{N+1} \widehat{L} \xrightarrow{\widehat{\sigma} \oplus \alpha^* \sigma} C^\infty(M, F \oplus F) \right), \quad \widehat{L} \in \widehat{\text{Even}}(M^{\text{odd}}), \\ & \frac{1}{2^{N+1}} \text{mod } 2^{N+1}\text{-ind} \left( 2^{N+1} \widehat{L} \xrightarrow{1 \oplus \alpha^* [\sigma^{-1}] \widehat{\sigma}} C^\infty(M, 2^N E) \right), \quad \widehat{L} \in \widehat{\text{Odd}}(M^{\text{ev}}). \end{aligned} \quad (17)$$

Applying the mod  $n$ -index formula, we obtain the desired topological expression

$$2^{N+1} \{d(\widehat{L})\} = p! [L]_N \in \mathbb{Z}_{2^{N+1}}, \quad [L]_N \in K(T^*M, \mathbb{Z}_{2^{N+1}}), \quad (18)$$

where  $[L]_N$  denotes the difference construction for the operators in (17)

$$[L] = \left[ 2^{N+1} L \xrightarrow{\sigma \oplus \alpha^* \sigma} \pi^* F \oplus \pi^* F \right] \quad \text{or} \quad \left[ 2^{N+1} L \xrightarrow{1 \oplus \alpha^* [\sigma^{-1}] \sigma} 2^N \pi^* E \right]. \quad (19)$$

We would like to rewrite (18) in a more canonical form.

To this end, consider the embedding

$$i : \mathbb{Z}_{2^{N+1}} \subset \mathbb{Z} \left[ \frac{1}{2} \right] / \mathbb{Z}, \quad i(x) = \frac{x}{2^{N+1}}.$$

It induces a mapping of  $K$ -groups

$$i_* : K(T^*M, \mathbb{Z}_{2^{N+1}}) \longrightarrow K\left(T^*M, \mathbb{Z} \left[ \frac{1}{2} \right] / \mathbb{Z}\right),$$

where the  $K$ -theory with dyadic coefficients is defined as the injective limit

$$K\left(T^*M, \mathbb{Z} \left[ \frac{1}{2} \right] / \mathbb{Z}\right) = \lim_{N' \rightarrow \infty} K(T^*M, \mathbb{Z}_{2^{N'}}). \quad (20)$$

**Lemma 1.** *The element  $[L] = i_*[L]_N \in K(T^*M, \mathbb{Z}[1/2]/\mathbb{Z})$  is well defined, i.e. independent of the choice of isomorphism  $\sigma$ .*

*Proof.* For two isomorphisms

$$2^N L \xrightarrow{\sigma} \pi^* F, \quad \text{and} \quad 2^N L \xrightarrow{\sigma'} \pi^* F'$$

let us compute the difference of the corresponding  $K$ -theory elements in (19). An explicit computation shows that the difference is equal to

$$\begin{aligned} [\sigma\sigma'^{-1}] \oplus \alpha^* [\sigma\sigma'^{-1}] & \quad \text{on an odd-dimensional manifold } M, \\ [\sigma\sigma'^{-1}] \oplus \alpha^* [\sigma'\sigma^{-1}] & \quad \text{on an even-dimensional } M. \end{aligned}$$

Thus, the difference in question is equal to

$$[\sigma_0] \pm [\alpha^* \sigma_0] \in K(T^*M)$$

for the elliptic symbol  $\sigma_0 = \sigma\sigma'^{-1}$ . The sign is opposite to the parity of  $\dim M$ .

To prove the Lemma, it suffices to show that  $\sigma_0$  defines a 2-torsion element in  $K(T^*M)$ . This is proved in the following purely topological Proposition.  $\square$

**Proposition 2.** *The antipodal involution  $\alpha : T^*M \longrightarrow T^*M$  induces an involution*

$$\alpha^* : K^*(T^*M) \longrightarrow K^*(T^*M)$$

*equal to  $(-1)^{\dim M}$ , modulo 2-torsion. More precisely, for an arbitrary  $x \in K(T^*M)$  and  $N$  large enough one has*

$$\alpha^*(2^N x) = (-1)^{\dim M} 2^N x. \quad (21)$$

*Proof.* The idea is to apply the Mayer–Vietoris principle.

1) Let us prove (21) for  $M = \mathbb{R}^n$ . We have

$$K^*(T^*\mathbb{R}^n) \simeq K^*(\mathbb{R}^{2n}) = \mathbb{Z},$$

while  $\alpha : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  acts as  $(x, \xi) \rightarrow (x, -\xi)$ . Thus, for  $n$  even it is homotopic to the identity and in the  $K$ -theory we have  $\alpha^* = id$ . While in odd-dimensions  $\alpha$  reverses the orientation. Therefore, in this case  $\alpha^* = -id$ , as desired.

2) We claim that the following assertion is valid: suppose that (21) is satisfied for two open subsets  $U, V \subset M$  and for their intersection  $U \cap V$ . Then these properties are valid for the union  $U \cup V$ .

A part of the Mayer-Vietoris exact sequence corresponding to the inclusions  $U \cap V \xrightarrow{i} U \sqcup V \xrightarrow{j} U \cup V$  is

$$\begin{array}{ccccc} K^{*+1}(T^*(U \cap V)) & \xrightarrow{\delta} & K^*(T^*(U \cup V)) & \xrightarrow{j^*} & K^*(T^*U) \oplus K^*(T^*V) \\ \downarrow \alpha^* & & \downarrow \alpha^* & & \downarrow \alpha^* \oplus \alpha^* \\ K^{*+1}(T^*(U \cap V)) & \xrightarrow{\delta} & K^*(T^*(U \cup V)) & \xrightarrow{j^*} & K^*(T^*U) \oplus K^*(T^*V). \end{array}$$

Suppose that the left and the right involutions in the diagram are equal to  $(-1)^{\dim M}$  (modulo 2-torsion). By a diagram chasing argument one deduces that the mapping  $\alpha^*$  in the center also satisfies (21). For example, on an even-dimensional manifold for  $x \in K^*(T^*(U \cup V))$  we get

$$j^*(\alpha^*x - x) = 0 \Rightarrow \alpha^*x - x = \delta\alpha^*y, \alpha^*y = y \Rightarrow 2(\alpha^*x - x) = 0$$

(in this computation factors  $2^N$  are omitted for brevity).

3) Consider a good (see [BT82]) finite covering  $\{U_\beta\}$  of the manifold  $M$  by contractible open sets. Over any  $U_\beta$  the property (21) is valid by the first part of the proof. Let us consider all subsets in  $\{U_\beta\}$ .

Passing from the coverings consisting of a single element to the covering of the entire manifold  $M$  and applying the assertion from the second part of the proof, we obtain the desired property for  $M$ .

Now Eq. (18) and Lemma 1 prove the following theorem.

**Theorem 5.** *A subspace  $\widehat{L} \in \widehat{\text{Even}}(M^{odd})$  or  $\widehat{\text{Odd}}(M^{ev})$  defines an element*

$$[L] \in K\left(T^*M, \mathbb{Z}\left[\frac{1}{2}\right] \Big/ \mathbb{Z}\right),$$

*and the fractional part of the functional  $d$  is computed by the direct image mapping*

$$\left\{d\left(\widehat{L}\right)\right\} = p_! [L] \in K\left(pt, \mathbb{Z}\left[\frac{1}{2}\right] \Big/ \mathbb{Z}\right) = \mathbb{Z}\left[\frac{1}{2}\right] \Big/ \mathbb{Z}$$

induced by  $p : M \rightarrow pt$ . In terms of an isomorphism  $\sigma : 2^N L \longrightarrow \pi^* F$  (see Proposition 1),  $[L]$  is defined by the symbol  $[(1 \pm \alpha^*) \sigma] \in K(T^*M, \mathbb{Z}_{2^{N+1}})$  as

$$[L] = i_* [(1 \pm \alpha^*) \sigma], \quad i : \mathbb{Z}_{2^{N+1}} \subset \mathbb{Z} \left[ \frac{1}{2} \right] / \mathbb{Z}$$

(the sign coincides with the parity of the subspace). A similar formula holds for the  $\eta$ -invariant

$$\{\eta(A)\} = p_![L_+(A)]$$

of an elliptic self-adjoint differential operator  $A$  satisfying the parity condition (1).

## 5 Examples and remarks

**1. Operators from [Gil89].** On a smooth oriented closed Riemannian odd-dimensional manifold  $M$ , we consider an elliptic self-adjoint differential operator of the second order

$$A = d\delta - \delta d : C^\infty(M, \Lambda^1(M)) \longrightarrow C^\infty(M, \Lambda^1(M)) \quad (22)$$

in the spaces of complexified 1-forms; here  $d$  is the exterior derivative and  $\delta$  is the adjoint operator with respect to the Riemannian metric. The principal symbol of  $A$  is

$$\sigma(A)(\xi) = \xi \wedge \xi] - \xi] \xi \wedge : \pi^* \Lambda^1(M) \longrightarrow \pi^* \Lambda^1(M),$$

where  $\xi \wedge$  is the exterior product by a covector  $\xi$  and  $\xi]$  is the inner product by the same covector with respect to the Riemannian metric. For an arbitrary point  $(x, \xi) \in S_x^* M$  of the cosphere bundle, the symbol  $L = L_+(A)$  of the spectral subspace  $\widehat{L} = \widehat{L}_+(A)$  coincides with the line spanned by the covector  $\xi$ . Hence,  $L \subset \pi^* \Lambda^1(M)$  is an even subbundle that at  $x \in M$  generates the reduced  $K$ -group of the projective space

$$[L] - [1] \in \widetilde{K}(P_x^* M) \simeq \mathbb{Z}_{2^{(\dim M - 1)/2}}, \quad \text{for } \dim M \geq 5.$$

Thus, the operator  $A$  defines a nontrivial element of  $K(P^* M)/K(M)$ . Let us compute the fractional part of the functional  $d$  on the subspace  $\widehat{L}_+(A)$ .

The line bundle  $L$  is trivial. We choose the trivialization

$$\begin{aligned} \sigma : L &\longrightarrow \pi^* \mathbb{C}, \\ \sigma(x, \xi) \eta &= \langle \xi, \eta \rangle_x, \end{aligned} \quad (x, \xi) \in S_x^* M, \quad \eta \in L_x, \quad (23)$$

where  $\langle \xi, \eta \rangle_x$  denotes the Hermitian inner product of two covectors with respect to the Riemannian metric at  $x$ . For the corresponding pseudodifferential operator

$$\widehat{\sigma} : \widehat{L} \longrightarrow C^\infty(M)$$

in the subspaces, the index formula (8) implies

$$\text{ind} \left( \widehat{\sigma}, \widehat{L}, C^\infty(M) \right) = \frac{1}{2} \text{ind} \widehat{\sigma} + d \left( \widehat{L} \right).$$

It follows from (23) that the symbol  $\widetilde{\sigma}$  is a direct sum of two constant symbols

$$\begin{aligned} \widetilde{\sigma} : \pi^* \Lambda^1(M) &\rightarrow \pi^* \Lambda^1(M), \\ \widetilde{\sigma}(\xi) &= \sigma^{-1}(-\xi) \sigma(\xi) \oplus 1 = -1 \oplus 1 : L \oplus L^\perp \rightarrow L \oplus L^\perp. \end{aligned}$$

Hence, we obtain the integrality result for the functional  $d$ :

$$d \left( \widehat{L} \right) = \text{ind} \left( \widehat{\sigma}, \widehat{L}, C^\infty(M) \right) \in \mathbb{Z}.$$

By the same method one can show the integrality of the functional for operators  $A$  with coefficients in a vector bundle  $E \in \text{Vect}(M)$ . To this end, one replaces the exterior derivative  $d$  and its adjoint by a covariant derivative  $\nabla$  and the corresponding adjoint operator for  $E$ . These operators were introduced in [Gil89], where the problem of nontriviality of their  $\eta$ -invariants was posed. Thus, we obtain

**Proposition 3.** *Operator  $d\delta - \delta d$  with coefficients in a bundle  $E$  has trivial fractional part of the  $\eta$ -invariant.*

As an application, consider the 3-dimensional real projective space  $\mathbb{RP}^3$ . It is parallelizable and the Kunneth formula shows that the group  $K(P^*\mathbb{RP}^3)/K(\mathbb{RP}^3)$  is generated by operators (22) with coefficients in vector bundles. Thus, this manifold has no even order operators with fractional  $\eta$ -invariants.

**2.** The vanishing result of Proposition 3 has an interesting corollary.

Assume  $M$  is odd-dimensional as before. Consider a pair  $S^*M \subset B^*M$ , where both spaces have the natural antipodal  $\mathbb{Z}_2$  action. The equivariant exact sequence of this pair leads to the following long exact sequence

$$\rightarrow K(M) \rightarrow K(P^*M)/K(M) \rightarrow K_{\mathbb{Z}_2}^1(T^*M) \rightarrow K^1(M) \rightarrow K^1(P^*M)/K^1(M) \rightarrow \dots \quad (24)$$

In analytic terms, the mapping  $K(M) \rightarrow K(P^*M)/K(M)$  corresponds to taking a bundle  $E$  to (symbol of) the operator  $d\delta - \delta d$  with coefficients in  $E$ .

Thus, the  $\eta$ -invariant as a mapping  $K(P^*M)/K(M) \rightarrow \mathbb{Z}[1/2]/\mathbb{Z}$ , is induced by a mapping of the subgroup in  $K_{\mathbb{Z}_2}^1(T^*M)$ . The elements of this latter equivariant  $K$ -group can be realized as symbols of indicial families in the sense of [Mel95] with a special symmetry. It seems that the  $\eta$ -invariant of Melrose would give the corresponding analytic realization of this mapping. However, this topic is beyond the scope of the present paper.

### 3. Orientable manifolds and the $\eta$ -invariant.

**Proposition 4.** *If  $M$  is orientable then the functional  $d$  and the  $\eta$ -invariant take integer, or half-integer values only.*

*Proof.* Indeed, Theorem 5 gives

$$2^{N+1}\{d(\widehat{L})\} = p_![(1 - (-1)^{\dim M} \alpha^*)\sigma] \in \mathbb{Z}_{2^{N+1}}.$$

Thus,

$$2^{N+2}\{d(\widehat{L})\} = 2p_!(1 - (-1)^{\dim M} \alpha^*)[\sigma \oplus \sigma] \in \mathbb{Z}_{2^{N+1}}, \quad [\sigma \oplus \sigma] \in K(T^*M, \mathbb{Z}_{2^{N+1}}).$$

However, M. Karoubi proved in [Kar70] that on an orientable  $M$  the antipodal involution  $\alpha$  acts as  $(-1)^{\dim M}$  in  $K$ -theory. Therefore, this yields

$$2^{N+2}\{d(\widehat{L})\} = p_!0 = 0.$$

Hence, we prove the desired  $2d(\widehat{L}) \in \mathbb{Z}$ . □

## 6 Elliptic theory with $\mathbb{Z}_n$ coefficients

The difference construction (13) is not an entirely computational trick involved in the modulo  $n$ -index calculation above. In the present section we show that similar to the usual difference construction, it establishes an isomorphism between the  $K$ -theory  $K(T^*M, \mathbb{Z}_n)$  with  $\mathbb{Z}_n$  coefficients and the group of stable homotopy classes of elliptic operators in subspaces of the form (11).

**1. Definition.** We consider elliptic operators of the form

$$D = n\widehat{L}_1 \oplus C^\infty(M, E_1) \longrightarrow n\widehat{L}_2 \oplus C^\infty(M, F_1), \quad (25)$$

where  $\widehat{L}_1 \subset C^\infty(M, E)$ ,  $\widehat{L}_2 \subset C^\infty(M, F)$ . This is slightly different from (11); the difference is motivated by the requirement that the inverse operator be of the same structure.

Let us state the stable homotopy classification problem for such operators. First, we introduce *trivial* operators. These are: a) operators induced by a vector bundle isomorphisms  $g : E_1 \rightarrow F_1$ :

$$C^\infty(M, E_1) \xrightarrow{g^*} C^\infty(M, F_1), \quad (26)$$

b) direct sums of  $n$  copies of an elliptic operator in subspaces:

$$n(\widehat{L}_1 \oplus C^\infty(M, E_1)) \xrightarrow{nD} n(\widehat{L}_2 \oplus C^\infty(M, F_1)). \quad (27)$$

We identify operators of the form (25) that differ by isomorphisms of the corresponding vector bundles  $E, F, E_1, F_1$ . Two elliptic operators  $D_1$  and  $D_2$  are *stably homotopic* if they become homotopic after we add some trivial operators to each of them. The abelian group formed by the classes of stably homotopic elliptic operators is denoted by  $\text{Ell}(M, \mathbb{Z}_n)$ . As usual, one can prove that the composition of elliptic operators  $D_{1,2}$  (if defined) gives an element  $[D_1 D_2]$  equal to the sum  $[D_1] + [D_2]$ .

**Lemma 2.** *An operator (25) is stably homotopic to an operator of the form*

$$n\widehat{L}' \xrightarrow{D'} C^\infty(M, F'). \quad (28)$$

*Proof.* The space  $C^\infty(M, E_1)$  can be eliminated in (25) by adding the trivial operator

$$(n-1) \left( C^\infty(M, E_1) \xrightarrow{id} C^\infty(M, E_1) \right).$$

The subspace  $\widehat{L}_2$  on the right-hand side of the formula can be eliminated in the following way. Let us add the trivial operator  $id : n\widehat{L}_2^\perp \rightarrow n\widehat{L}_2^\perp$  to  $D$ . Then we obtain an operator of the form

$$n \left( \widehat{L}_1 \oplus \widehat{L}_2^\perp \right) \longrightarrow n \left( \widehat{L}_2 \oplus \widehat{L}_2^\perp \right) \oplus C^\infty(M, F_1).$$

To complete the proof, it suffices to show that the resulting subspace

$$\widehat{L}_2 \oplus \widehat{L}_2^\perp \subset C^\infty(M, F_1) \oplus C^\infty(M, F_1)$$

is homotopic to the subspace  $C^\infty(M, F_1) \oplus 0$ , since a homotopy of subspaces can be lifted to a homotopy of elliptic operators. The desired homotopy of subspaces is given in terms of the projection  $P$  on the subspace  $\widehat{L}_2$  by the formula

$$\widehat{L}_\varphi = \text{Im } P_\varphi, \quad P_\varphi = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} + (1 - P) \begin{pmatrix} \sin^2 \varphi & \cos \varphi \sin \varphi \\ \cos \varphi \sin \varphi & \cos^2 \varphi \end{pmatrix}.$$

Here  $\widehat{L}_\varphi \subset C^\infty(M, F_1) \oplus C^\infty(M, F_1)$  and  $\varphi$  varies from 0 to  $\pi/2$ .  $\square$

**2. Exact sequence in Elliptic theory.** Denote by  $\text{Ell}(M)$  the group of stable homotopy classes of elliptic operators on  $M$ . Let  $\text{Ell}_1(M)$  denote a similar group of stable homotopy classes of pseudodifferential subspaces. More precisely, two subspaces are called homotopic, if there is a norm continuous homotopy of projections defining them. They are stably homotopic, if they become homotopic if we add some *trivial subspaces* to them. Here the trivial subspaces are spaces of sections of vector bundles and finite-dimensional spaces.



Let us now define a sequence

$$\mathrm{Ell}(M) \xrightarrow{\times n} \mathrm{Ell}(M) \xrightarrow{i} \mathrm{Ell}(M, \mathbb{Z}_n) \xrightarrow{j} \mathrm{Ell}_1(M) \xrightarrow{\times n} \mathrm{Ell}_1(M), \quad (29)$$

where  $\times n$  denotes multiplication by  $n$ , the mapping  $i$  is induced by the natural inclusion of the usual elliptic operators in the  $\mathrm{mod} n$ -theory, and the boundary mapping  $j$  is defined as

$$j \left[ n\widehat{L}_1 \oplus C^\infty(M, E_1) \xrightarrow{D} n\widehat{L}_2 \oplus C^\infty(M, F_1) \right] = [\widehat{L}_1] - [\widehat{L}_2].$$

**Proposition 5.** The sequence (29) is exact.

*Proof.* It is straightforward to show that (29) is a complex. Let us prove the exactness.

Let  $[D] \in \ker j$ . By Lemma 2 we can suppose that  $D$  has the form (28). Since  $[\widehat{L}'] = 0 \in \mathrm{Ell}_1(M)$ , it follows that the subspace  $\widehat{L}'$  is homotopic<sup>3</sup> to a space of sections of a vector bundle. Hence,  $D$  is homotopic to an elliptic operator in the spaces of vector bundle sections. Consequently, we obtain  $[D] \in \mathrm{Im} i$ .

Let  $[\widehat{L}] \in \ker \{\times n\}$ . This implies that the subspace  $n\widehat{L}$  is homotopic to the space of sections of a vector bundle. Consequently, there exists an elliptic operator  $D : n\widehat{L} \rightarrow C^\infty(M, F)$ . Hence,  $[\widehat{L}] = j[D]$ , as desired. The remaining assertion  $[D] \in \ker i \Rightarrow [D] \in \mathrm{Im}(\times n)$  can be proved along similar lines and is left to the reader.  $\square$

**3. Isomorphism of Elliptic theory and  $K$ -theory.** By virtue of Lemma 28, we can extend the difference construction

$$D \longmapsto [\sigma(D)] \in K(T^*M, \mathbb{Z}_n) \quad (30)$$

(see (13)) to a homomorphism of groups

$$\mathrm{Ell}(M, \mathbb{Z}_n) \longrightarrow K(T^*M, \mathbb{Z}_n),$$

since the mapping (13) sends the trivial operators (26) and (27) to zero.

Let us also recall the difference construction for pseudodifferential subspaces. Namely, a subspace  $\widehat{L} = \mathrm{Im} P$  with symbol  $L = \mathrm{Im} \sigma(P)$  defines a family of elliptic symbols on  $M$  with the parameter space  $\mathbb{S}^1$  and coordinate  $z$ :

$$z\sigma(P) + (1 - \sigma(P)) : \pi^*E \longrightarrow \pi^*E.$$

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<sup>3</sup>Here and in what follows we omit the standard considerations concerning the stabilization of elements.

By virtue of the usual difference construction for elliptic families, this defines the desired element (see [APS76])

$$[z\sigma(P) + (1 - \sigma(P))] \in K(T^*M \times \mathbb{S}^1, T^*M \times pt) \equiv K^1(T^*M). \quad (31)$$

Let us now consider the following diagram

$$\begin{array}{ccccccccc} \text{Ell}(M) & \xrightarrow{\times n} & \text{Ell}(M) & \xrightarrow{i} & \text{Ell}(M, \mathbb{Z}_n) & \xrightarrow{j} & \text{Ell}_1(M) & \xrightarrow{\times n} & \text{Ell}_1(M) \\ \downarrow \chi_0 & & \downarrow \chi_0 & & \downarrow \chi_n & & \downarrow \chi_1 & & \downarrow \chi_1 \\ K(T^*M) & \xrightarrow{\times n} & K(T^*M) & \xrightarrow{i'} & K(T^*M, \mathbb{Z}_n) & \xrightarrow{j'} & K^1(T^*M) & \xrightarrow{\times n} & K^1(T^*M), \end{array} \quad (32)$$

where  $\chi$  with subscripts denote difference constructions, and the lower row in the diagram is part of the exact coefficient sequence in  $K$ -theory (see Appendix).

**Lemma 3.** *The diagram (32) is commutative.*

*Proof.* The commutativity of the leftmost and rightmost squares of the diagram is clear. Let us consider the second square:

$$\begin{array}{ccc} \text{Ell}(M) & \xrightarrow{i} & \text{Ell}(M, \mathbb{Z}_n) \\ \downarrow \chi_0 & & \downarrow \chi_n \\ K(T^*M) & \xrightarrow{i'} & K(T^*M, \mathbb{Z}_n). \end{array}$$

For an elliptic operator  $D$  it is easy to see that

$$\chi_n i[D] = [\sigma(D)]([\gamma_n] - 1) \in K(T^*M \times \mathbb{M}_n, T^*M \times pt);$$

here  $[\sigma(D)] = \chi_0[D] \in K(T^*M)$  is the usual difference construction. On the other hand, the reduction modulo  $n$  mapping  $i'$  is exactly the multiplication by the element  $[\gamma_n] - 1$ . Thus, the second square in (32) is commutative.

Finally, let us check the commutativity of the remaining third square

$$\begin{array}{ccc} \text{Ell}(M, \mathbb{Z}_n) & \xrightarrow{j} & \text{Ell}_1(M) \\ \downarrow \chi_n & & \downarrow \chi_1 \\ K(T^*M, \mathbb{Z}_n) & \xrightarrow{j'} & K^1(T^*M). \end{array}$$

For an elliptic operator  $D : n\widehat{L} \rightarrow C^\infty(M, F)$ , on the one hand, we obtain

$$\chi_1(j[D]) = [L] \in K^1(T^*M).$$

On the other hand, the difference construction for  $D$  gives

$$\chi_n[D] = \left[ \pi^* F \xrightarrow{\sigma^{-1}(D)} nL \xrightarrow{\beta^{-1} \otimes 1_L} \gamma_n \otimes nL \xrightarrow{1_\gamma \otimes \sigma(D)} \gamma_n \otimes \pi^* F \right] \in K(T^*M, \mathbb{Z}_n). \quad (33)$$

In terms of the identifications

$$\begin{aligned} K(T^*M, \mathbb{Z}_n) &= K(T^*M \times \mathbb{M}_n, T^*M \times pt), \\ K^1(T^*M) &= K(T^*M \times \mathbb{S}^1, T^*M \times pt), \end{aligned}$$

the Bokstein mapping  $j'$  is induced by the inclusion  $\mathbb{S}^1 \xrightarrow{i_0} \mathbb{M}_n$ . More precisely,

$$j' = (1_{T^*M} \times i_0)^*.$$

Let us compute the family of elliptic symbols in (33) on the circle  $\mathbb{S}^1 \subset \mathbb{M}_n$  with a polar coordinate  $\zeta = e^{i\varphi}$ ,  $0 \leq \varphi < 2\pi$ . The family in (33) has the form

$$\pi^* F \xrightarrow{\sigma^{-1}} nL \xrightarrow{\zeta \oplus 1_{n-1}} nL \xrightarrow{\sigma} \pi^* F \quad (34)$$

with respect to the natural trivialization of  $\gamma_n$  on  $\mathbb{S}^1$ . Here the principal symbol of  $D$  is denoted by  $\sigma$  and the diagonal operator  $\zeta \oplus 1$  acts as  $(\zeta \oplus 1)(u_1, u_2, \dots, u_n) = (\zeta u_1, u_2, \dots, u_n)$ .

Let us also rewrite the symbol  $\sigma$  in block matrix form:

$$\sigma = (\sigma_1, \dots, \sigma_n), \quad \sigma_i : L \longrightarrow \pi^* F.$$

The ellipticity of  $\sigma$  implies that the components  $\sigma_i$  are monomorphic. Consider also the inverse symbol  $\sigma^{-1}$

$$\sigma^{-1} = (\sigma^1, \dots, \sigma^n)^t, \quad \sigma^i : \pi^* F \longrightarrow L.$$

We readily obtain

$$\sum_{i=1}^n \sigma_i \sigma^i = 1, \quad \sigma^i \sigma_j = \delta_j^i.$$

Hence,  $\sigma_1 \sigma^1$  is the projection on a subbundle isomorphic to the original bundle  $L$

$$\text{Im } \sigma_1 \sigma^1 \xrightarrow{\sigma_1^1} L.$$

Therefore, the family (34) defines an element

$$[\zeta \sigma_1 \sigma^1 + (1 - \sigma_1 \sigma^1)] \in K(T^*M \times \mathbb{S}^1, T^*M \times pt) = K^1(T^*M).$$

This element coincides with the difference construction for  $\widehat{L}$  (see (31)).  $\square$

**Theorem 6.** *The difference construction*

$$\mathrm{Ell}(M, \mathbb{Z}_n) \xrightarrow{\chi_n} K(T^*M, \mathbb{Z}_n)$$

*is an isomorphism.*

*Proof.* The usual difference constructions

$$\mathrm{Ell}(M) \xrightarrow{\chi_0} K(T^*M) \quad \text{and} \quad \mathrm{Ell}_1(M) \xrightarrow{\chi_1} K^1(T^*M)$$

are isomorphisms, so the theorem is proved by applying the 5-lemma to the commutative diagram (32).  $\square$

## Appendix. $K$ -theory with coefficients

Here we recall some basic properties of the  $K$ -theory with  $\mathbb{Z}_n$  coefficients that are used in the present paper. More details can be found, e.g. in [AT65, Bla98], and the references therein. By  $n$  we denote a positive integer,  $n \geq 2$ .

**1. Moore space.** Let us consider the 2-dimensional complex  $\mathbb{M}_n$  obtained from the unit disk  $D^2$  identifying points on its boundary under the  $\mathbb{Z}_n$  action:

$$\mathbb{M}_n = \{ D^2 \subset \mathbb{C} \mid |z| \leq 1 \} / \left\{ e^{i\varphi} \sim e^{i(\varphi + \frac{2\pi k}{n})} \right\}.$$

The result is called the *Moore space*. For instance,  $\mathbb{M}_2 = \mathbb{RP}^2$ . There is an embedded circle  $\mathbb{S}^1$  in the Moore space:

$$\mathbb{S}^1 = \left\{ e^{i\varphi} \mid 0 \leq \varphi \leq \frac{2\pi}{n} \right\} \subset \mathbb{M}_n,$$

while the quotient space is homeomorphic to the 2-sphere  $\mathbb{M}_n/\mathbb{S}^1 = \mathbb{S}^2$ . The exact sequence of the pair  $(\mathbb{M}_n, \mathbb{S}^1)$  in  $K$ -theory reduces to

$$\begin{array}{ccccccc} 0 \rightarrow & K^1(\mathbb{S}^1) & \xrightarrow{\delta} & \tilde{K}(\mathbb{S}^2) & \rightarrow & \tilde{K}(\mathbb{M}_n) & \rightarrow 0 \\ & \parallel & & \parallel & & & \\ & \mathbb{Z} & & \mathbb{Z} & & & \end{array}$$

and the coboundary mapping  $K^1(\mathbb{S}^1) \xrightarrow{\delta} \tilde{K}(\mathbb{S}^2)$  acts as the multiplication by  $n$ . This description gives

$$K^1(\mathbb{M}_n) = 0, K(\mathbb{M}_n) = \mathbb{Z} \oplus \mathbb{Z}_n.$$

The generator of the torsion part  $\mathbb{Z}_n$  is  $[\gamma_n] - 1 \in K(\mathbb{M}_n)$ , where  $\gamma_n \in \mathrm{Vect}(\mathbb{M}_n)$  is the pull-back of the Hopf line bundle on  $\mathbb{S}^2$ .

The Whitney sum  $n\gamma_n$  is a trivial vector bundle. In fact, its transition function is equal to  $(z, z, \dots, z)$ . It is homotopic to  $(z^n, 1, \dots, 1)$ . Using the latter transition function, it is easy to produce a trivialization  $n\gamma_n \xrightarrow{\beta} \mathbb{C}^n$ .

**2.  $K$ -groups with coefficients.** For a topological space  $X$ , the  $K$ -theory with coefficients  $\mathbb{Z}_n$  is defined in terms of the usual (integral)  $K$ -theory by the formula

$$K^*(X, \mathbb{Z}_n) = K^*(X \times \mathbb{M}_n, X \times pt). \quad (35)$$

For instance, for a point we have

$$K^*(pt, \mathbb{Z}_n) = \tilde{K}^*(\mathbb{M}_n),$$

which is trivial for  $K^1$  and  $\mathbb{Z}_n$  for  $K^1$ .

There is an exact sequence in  $K$ -theory with coefficients

$$\rightarrow K(X) \xrightarrow{\times n} K(X) \rightarrow K(X, \mathbb{Z}_n) \rightarrow K^1(X) \xrightarrow{\times n} K^1(X) \rightarrow K^1(X, \mathbb{Z}_n) \rightarrow, \quad (36)$$

corresponding to the short exact sequence  $0 \rightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0$ . It is obtained from the exact sequence of the pair  $(X \times \mathbb{M}_n, X \times \mathbb{S}^1)$  by the Bott periodicity. We will need explicit descriptions of the connecting homomorphisms. The "reduction modulo  $n$ " mappings

$$K(X) \rightarrow K(X, \mathbb{Z}_n) \quad \text{and} \quad K^1(X) \rightarrow K^1(X, \mathbb{Z}_n)$$

are realized as tensor products with  $[\gamma_n] - 1 \in \tilde{K}(\mathbb{M}_n)$ , while the Bokstein maps

$$K(X, \mathbb{Z}_n) \rightarrow K^1(X) \quad \text{and} \quad K^1(X, \mathbb{Z}_n) \rightarrow K(X)$$

are induced by the embedding  $\mathbb{S}^1 \subset \mathbb{M}_n$ .

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